

Lecture no. 3, Professor S.T. Yau , April 10, 2007

Notes and supplementary comments (in []s) by Robert E. Greene

Last time: Real 4-manifolds M^4 with almost (many) complex structures but with no integrable almost complex structure, no complex structure. In understanding this situation, we observed the importance of the Riemann-Roch –Hirzebruch formula, and of the fact that it actually holds in the non-Kähler case (from the Atiyah-Singer Theorem). Namely for a compact complex manifold of any dimension and with F the sheaf of germs of sections of a holomorphic vector bundle V :

$$\sum (-1)^i \dim H^i(M, F) = \int ch(V) Todd(M)$$

where ch is the Chern character and Todd is the Todd class.

In the case of a compact complex manifold M of complex dimension 2, and with $V = \mathcal{O}$ the trivial line bundle, one gets

the left hand side = the "arithmetic genus" (by definition) while

$$\text{the right hand side} = (c_1^2 + c_2)/12$$

Definition: irregularity of M , symbol q , is $\dim H^1(M, \mathcal{O})$ (that is, the middle item in the left hand side).

Kodaira proved from the Riemann Roch formula that

$2q - b_1 = 1$ or 0 (in this complex 2-dimensional case), where b_1 is the first Betti number as usual.

This cannot be 1 in case M is Kähler, since as we shall now recall, b_1 is necessarily even in the Kähler case (in all dimensions). [This is of course a generalization of the fact that the first Betti number of a compact Riemann surface is always even, namely $2g$ with $g =$ the topological genus of the surface].

For this, define as usual the Hodge numbers $h^{p,q}$ to be the dimension of the (Cech) cohomology group $H^q(M, \Omega^p)$, where Ω^p is the holomorphic vector bundle of forms of type $(0,q)$ [In other words, $h^{p,q}$ is the dimension of the p th sheaf cohomology of the sheaf of germs of sections of the sections of the holomorphic vector bundle Ω^p . This is by the Dolbeault isomorphism given by $\bar{\partial}$ q -cohomology for forms with values in Ω^p .] In the Kähler case this is given by the dimension of a space of harmonic forms, namely the dimension of the space of harmonic forms of type (p,q) . Here as usual type (p,q) means that in a holomorphic local coordinate system they have the form

$$\sum f_{I,J} dz^I \wedge \bar{d}\bar{z}^J \quad \text{where } I \text{ and } J \text{ are index sets of length } p \text{ and } q, \text{ respectively.}$$

Now the ordinary Laplacian acting on C^∞ m -forms (with complex coefficients) has a (finite dimensional) kernel, which consists of the harmonic forms by definition and has dimension $= b_m$. On any complex manifold, one has projection operators taking (pointwise) the space of m -forms to the space of (p,q) forms,

$$\prod_{p,q} m\text{-forms} \rightarrow (p,q) \text{ forms.}$$

On a Kähler manifold, these operators commute with the Laplacian. This follows essentially from the fact that the J operator and hence the Kähler form is a covariant constant (parallel relative to the Riemannian connection) and the Laplacian is essentially constructed from the Kähler form in this case.

This gives the famous "Hodge decomposition"

$$\ker \Delta \text{ on smooth } m\text{-forms} = \bigoplus_{p+q=k} (\ker \Delta \text{ on } (p,q) \text{ forms}).$$

$$\text{So } b_m = \sum h^{p,q}.$$

Also $h^{p,q} = h^{q,p}$ since conjugation takes the kernel of the Laplacian on (p,q) forms to the kernel on (q,p) forms, the Laplacian being itself a real operator.

In particular, $h^{1,0} = h^{0,1}$ and $b_1 = h^{1,0} + h^{0,1}$ so $b_1 = 2 h^{1,0}$ is even, as we sought.

Philosophically, the important point here is that b_1 is of course purely topological but this purely topological item gives information about the holomorphic items $h^{p,q}$ and thus potentially about spaces of holomorphic sections.

Note that, in effect because the Riemann-Roch theorem still works in the non-Kähler case, one still gets information about the irregularity from topological information, even in the non-Kähler case.

Holomorphic 1-forms on complex surfaces and why they are important:

Suppose M is a compact complex surface, not necessarily Kähler. And suppose that $\omega = \sum f_i dz_i$ is a holomorphic 1-form. Then the 1-form must be closed.

This is surprising at first sight but easy to prove: Since $d\bar{\omega}$ vanishes on the 1-form (because it is holomorphic), $d\omega$ is a $(2,0)$ form and hence can be written

$$F dz_1 \wedge dz_2.$$

Now by Stokes Theorem $\int d\omega \wedge d\bar{\omega} = 0$ (because $d(\omega \wedge \bar{\omega}) = d\omega \wedge d\bar{\omega}$).

On the other hand, $\int d\omega \wedge d\bar{\omega} = \int |F|^2$. So F is identically 0 and ω is closed.

With a fixed point z_0 chosen in M once and for all, then for each (necessarily closed) holomorphic form ω , point z in M , and path from z_0 to z , one can associate the integral of ω along the path. This is well-defined up to an integral of the form along a closed path in M . So one can associate to each z in M a linear functional on the space of holomorphic (closed) $(1,0)$ forms and this is well defined up to the "periods", namely (since the forms are closed) the integrals of the forms over the 1-homology of M with integer coefficients. Tracing through this one gets the "Albanese map" of M into a torus, namely the dual space of the holomorphic $(1,0)$ forms / image $H_1(M, \mathbb{Z})$, where $H_1(M, \mathbb{Z})$ is mapped into the dual space of the holomorphic $(1,0)$ forms by integration. This map is holomorphic. [Note: The dimension of the dual space of the space of holomorphic $(1,0)$ forms is of course the same as the (finite) dimension of the space of holomorphic 1-forms themselves, but the passage to the dual is important since otherwise the exact role of the periods is obscured.]

The Albanese map's very definition depends on the forms being closed: otherwise the integration depends on the path in a way much more complicated than "well defined mod periods". Without closed, it is hopeless!

The dimension of the "Albanese torus" is the irregularity q , and the Albanese map is thus nontrivial if $q > 0$. And as noted earlier, this is guaranteed by Kodaira's formula even in the non-Kähler case, if one assumes that $b_1 > 0$. The nontriviality of the Albanese map in turn gives one information about the surface, since the image is Kähler and the fibre is a curve or a point. Yau(1976, Topology) used this method to exhibit parallelizable 4-manifolds without complex structures. These had b_1 equal to 4 and hence if there had been a complex structure one would have had the Albanese map going to a 3-torus (at least), but this could be observed to be impossible in the particular case. This illustrates the importance of the idea of using the purely topological b_1 in this instance, to get holomorphic objects.

There are really only two methods so far for exhibiting almost complex manifolds with no complex structure. In historical order;

1 Van de Ven : Chern number restrictions

and

2 the holomorphic 1-form argument of the type just discussed

For method 1, in van de Ven [Proc. Nat. Acad. Sci, 1966, discussed in detail in Compact Complex Surfaces, Barth, Peters, Van de Ven, Springer 1984], it was shown that for a compact complex surface $8c_2 \geq c_1^2$ [It is also shown how to find almost complex manifolds with c_1 and c_2 having arbitrary values subject only to the condition that 12 divides $c_1^2 + c_2$. Of course infinitely many of these violate the inequality shown. Van de Ven also provides some other examples using an Albanese torus argument similar in spirit to that of method 2, though not giving the trivial tangent bundle case shown by Yau. Kodaira's classification is used here.]

The Van de Ven inequality was improved by Bogomolov to $4 c_2 \geq c_1^2$ and subsequently by Miyoka and independently Yau to $3 c_2 \geq c_1^2$. This latter is optimal as one sees from $\mathbb{C}P^2$ with $c_2 = 3$ and $c_1^2 = 9$. [The first is just the Euler characteristic. For the second, note that the first Chern class here is that of the dual of the canonical bundle and that the canonical bundle's dual is the third power of the dual of the tautological bundle, so that in terms of intersection numbers one is looking for the square of this at the intersection of three hyperplanes (complex lines in this case) in $\mathbb{C}P^2$ with three others in general position for a total of nine intersection points, all of course of positive orientation. Note that sign conventions here on the first Chern class are irrelevant since the first Chern class is squared.]

Higher dimensions: Unfortunately, all of this is specific to complex dimension 2 and does not generalize in any apparent way to higher dimensions. In particular, in the higher dimensional Riemann-Roch there are at least two "middle terms" and these occur with opposite sign. So one does not easily get any inequality that implies (as in the Kodaira irregularity formula) the existence of holomorphic objects.

The most natural next stage is of course to look at M^6 , a compact manifold of real dimension 6 with an almost complex structure. And then one asks as before if there is always an integrable one. No example has been found where it can be seen that such an M exists with demonstrably no complex structure. So it is natural to conjecture that in real dimension 6, the existence of an almost complex structure implies the existence of a (complex dimension 3) complex structure.

The famous instance is S^6 . But it is probably not a good idea to concentrate on this specific instance rather than looking for a general method. Calabi showed that there is an almost complex structure coming from G_2 on any compact hypersurface in \mathbb{R}^7 . [Calabi, Construction and properties of some six-dimensional almost complex manifolds, *Transactons A.M.S.* 87(1958)]. So S^6 is rather special already. Most of the Calabi structures are non-Kähler and are not themselves integrable but the question remains if there is an integrable structure.

Return to our earlier concept: an almost complex structure is a lift of the classifying map of M into $BSO(6)$ for the tangent bundle to $BU(3)$. Then one can hope to deform such a lift, which is itself an almost complex structure, to an integrable almost complex structure. But one suspects that this may not be possible, in the same general way that maps of S^2 into a given manifold may not be deformable to minimal ones because one may encounter the "bubbling" phenomenon. The original S^2 shrinks may divide into two pieces, if it in some sense goes around two "holes" in the manifold. This was analyzed by Sachs-Uhlenbeck. [This has already played a role, e.g., in Siu/Yau on the Frankel Conjecture and Micallef-Moore.]

In higher dimensions, one can also have singularities. Similar bubbling phenomena (and possibly singularities in high dimensions) are likely to happen in the almost complex deforming to complex structure question, e.g., on M^6 real dimension 6 almost complex manifolds in trying to deform to a 3-complex-dimension complex structure.

But one should shift focus to general M , not just the 6-sphere. This is philosophically similar to the (three dimensional) Poincare Conjecture, where concentration on the three-sphere as such was not effective whereas the approach to general geometrization gave the key ideas for the sphere problem in particular. Some similar principle probably applies for the S^6 problem: the key will probably be a general method.

Complex Surfaces Revisited.

We now return to complex surface theory to look for some ideas that might be generalizable to higher dimensions. In complex dimension 2, we have the famous Kodaira classification., which grew out of the work of the Italian algebraic geometers(Castelnuovo, Enriques,et. al.)

[The Kodaira/Enriques classification arises as a generalization of the Riemann surface situation. One first looks for a way to distinguish the three cases , genus 0, genus 1 , and higher genus in terms of algebraic geometric ideas alone. (It is futile to expect the constant curvatures corresponding , positive , zero, and negative to work in higher dimensions as such!). Consider the canonical bundle K of $(1,0)$ forms. This is a negative bundle on CP^1 and so all its positive power are also negative, and the space of holomorphic sections of K^m , $m > 0$, is of dimension 0 for all $m > 0$. On a torus(genus 0), K is holomorphically trivial, as are all its (positive) power consequently, and the dimension of the space of holomorphic sections of K^m is 1 for all $m(>0)$. On a higher genus Riemann surface Σ , K is a positive bundle(the degree of the bundle is $2g-2$) and hence so are its positive powers. And the Riemann Roch Theorem gives that the dimension of the space of holomorphic sections $H^0(M, K^m)$ is $m(2g-2) + 1 - g + \dim H^0(M, K^{1-m})$. If $m > 1$, then K^{1-m} is a negative bundle and H^0 has dimension 0. (Recall the $3g - 3$ dimension for quadratic differentials!) So the dimension of the space of sections of K^m , $m > 0$, grows linearly as a function of m . This suggests that one should consider for higher dimensional complex manifolds, the behavior of $\dim H^0(M, K^m)$ for positive m large, that is the asymptotic behavior of this dimension. This is the idea that underlies the Kodaira/Enriques classification.]

On a complex surface, there is a natural holomorphic line bundle, the "canonical bundle" of forms of type $(n,0)$. As discussed earlier, the transition functions here are the holomorphic Jacobian of coordinate changes, that is the determinant of the $n \times n$ matrix of partial derivative of w_j with respect to z_i . Then we can look at the (positive) powers of these transition functions, that is the m th powers of this (holomorphic on coordinate overlaps) determinant. This gives the holomorphic line bundle K^m .

By definition, $\dim H^0(M, K^m)$ is the m th "pluri-genus" p_m and the sequence p_1, \dots are called collectively the "plurigenera". The sequence of plurigenera together form a powerful invariant of the complex surface. [The plurigenera are "birational invariants" so in particular, they do not change "under blowing up" or down, as opposed to say $h^{1,1}$. Literally, a birational invariant of an algebraic variety is something that is determined by the meromorphic function field and is thus the same for all varieties with that function field.]

We look now for a classification of "minimal models". A minimal model is a surface where no (rational) curve that can be blown down (collapsed to a point). Theorem of Grauert: If $P^1 \rightarrow M$ has self-intersection number -1 then there is an M' with a point p' such that $M = M'$ with p blown up and the blow up of p' is the curve P . From this, one deduces that by successively blowing down such "exceptional curves" one arrives at a situation where no further blowing down is possible, a "minimal model". [Minimal models of algebraic surfaces are unique unless the surface is "ruled", A ruled surface is a surface that is birational to the product of a P^1 and a Riemann surface of genus g at least 1.]

So in summary for complex surfaces, any surface is obtained from a minimal model (with no exceptional curves that can be blown down) via a finite number of successive blow ups.

Now we turn to the Kodaira classification: this is a list of possibilities, some with sub-cases.

(I) All plurigenra are 0. This corresponds to Kodaira dimension $= -\infty$.

[Kodaira dimension in general is defined as follows: On a compact complex manifold, either all but a finite number of the plurigenra are 0 or there is a unique integer k such that for some positive numbers A and B , $A m^k \leq p_m \leq B m^k$. The integer k is called the Kodaira dimension of M . It is always no larger than the complex dimension of M . In the first case, that all but a finite number of plurigenra are 0, one says the Kodaira dimension is $-\infty$ (or sometimes -1). The Kodaira dimension is equal to the maximal dimension of the mapping of M into complex projective space via sections of a power of the canonical bundle, that is, the mapping that becomes an embedding in the proof of the Kodaira Embedding Theorem via sections of positive bundles. Thus the Kodaira dimension of a manifold of complex dimension n with positive canonical bundle is n , the maximum possible value. But in general the Kodaira dimension can be less than n . For example, in our Riemann surface motivational case, the Kodaira dimension for CP^1 is $-\infty$ (plurigenra all 0); for a genus 0 surface (a torus), it is 0 (plurigenra are all 1); and for a surface of genus bigger than 1, the Kodaira dimension is 1, linear growth of plurigenra as already noted.]

This class (I) includes

1. "rational surfaces" (birational to CP^2), with minimal model CP^2 itself.
2. the Hirzebruch surfaces (except for the $n=1$ case), which are P^1 bundles over P^1 and ruled surfaces, which are P^1 bundles over a Riemann surface Σ of genus g greater than 1.
3. finally there are the mysterious Kodaira VII surfaces, which have not been really classified (more on this later; these have $b_1=1$).

II Kodaira dimension $=0$

Here $p_m = 1$ for some m and $p_m < 2$ for all m .

Classes of these(continuing the consecutive numbering)

4. K3 surfaces, with irregularity =0, $p_1=1$ and canonical line bundle trivial. All K3s are diffeomorphic in the real sense to the quartic surface $\sum z_j^4 = 0$ in $\mathbb{C}P^3$.

5. Enriques surface which has $q=0$, but canonical bundle not trivial. This is K3/ Z_2 for some Z_2 action

6. complex torus(including abelian varieties)[In complex dimension 2 and higher, complex tori, that is quotients of \mathbb{C}^n by a lattice of translations, are generically not algebraic. This was discovered by Riemann. It is the starting point for the question of how to characterize algebraic varieties among Kähler manifolds—the tori are always Kähler since they inherit the Euclidean metric—which culminated in the idea of a Hodge metric and Kodaira's proof that a Kähler manifold was algebraic if (and only if) it admitted a Hodge metric.]

7. hyperelliptic surfaces [These are quotients of the product of two elliptic curves(Riemann surfaces of genus 0) by finite group actions. There are seven families of these corresponding to various possibilities for the group involved.]

8. Kodaira surfaces: nonalgebraic, but with some nonconstant meromorphic functions. Two kinds: (i) $b_1=3$ and (ii) $b_1=1$ [the first kind have trivial canonical bundle, the second kind nontrivial. The second kind are quotients of the first kind by finite groups of small order, 2,3,4, or 6. For the first kind, the plurigenra are all 1 since the canonical bundle is trivial. For the second kind, the plurigenra are all 0 except if the index is divisible by the order of the group in which cases they are 1.]

These surfaces (of first type) arise from quotients of \mathbb{C}^2 by groups of affine transformations of determinant 1. There is a global holomorphic 2-form ($dz_1 dz_2$ pushes down to quotient). They can also be represented as elliptic fibrations over an elliptic curve with constant moduli on the fibre(usually fibre varies).

III Kodaira dimension=1

9. (proper) elliptic surfaces: these are fibered over a Riemann surface of genus at least 2 with fibre an elliptic curve with variable moduli (fibre a Riemann surface of genus 0 but with possibly varying complex structure) and possible singular fibres. The singular fibres that could occur were classified by Kodaira. [These all have $c_1^2=0$ and c_2 nonnegative.]

IV Kodaira dimension =2

10. These are all regarded as one group. [As noted, they are the analogue of Riemann surfaces of genus at least 2. They are all algebraic. Both c_1^2 and c_2 are positive and their sum is divisible by 12.]

End of classification

Looking at this, one learns a lot of general facts about complex surfaces. First of all, being Kähler seems quite close to being algebraic in some sense. Kodaira actually proved that every Kähler surface can be deformed into an algebraic surface (Castelnuovo conjecture) [e.g., turn any complex torus into an abelian variety by deforming lattice]. Deformation here means a homomorphic family of surfaces. So topologically there is no difference between the two.

For a long time this was believed to be true in higher dimensions as well, But Claire Voisin showed that this is not the case: in every dimension from 3 on up, there are Kähler manifolds which cannot be deformed to be algebraic.

Second question: Non-Kähler versus Kähler.

Kähler implies b_1 is even as before. For surfaces the converse is true: b_1 even implies surface is Kähler. The argument that follows depends on the classification, but a direct proof without classification is possible. (Buchdahl; for next time)

The argument works from the classification. The only cases to worry about are K3 surfaces and surfaces of Kodaira dimension 1. The rest are either algebraic or are quotients of algebraic by a discrete group. In particular, group 8 are ruled out because of Betti numbers 1 or 3.

Elliptic surfaces case was proved by Miyoka (Japanese Acad Proc, 50(1974))

Surfaces of type VII (item 3 in the Class I of the previous) are very interesting non-Kähler examples. There are two types, one type with $b_2=0$, the other with b_2 positive. Inoue proved constructed examples which contain no curves. J. Li, S.T. Yau and F. Zheng completed the proof (clarifying an early work of Bogomolov) that a class VII surface with no curves and $b_2=0$ was one of Inoue's examples. Kodaira had already classified those of class VII with $b_2=0$ and at least one curve—these must be either elliptic or a Hopf surface. Thus the $b_2=0$ situation is completely understood.

Later, in 1983, Harvey and Lawson proved the elliptic surface case by another method involving geometric measure theory. (Inventiones Math. 74). [The statement and proof of their basic result involves the concept of a current. A current is an element in the dual space of forms of a certain degree, in analogy to distributions as the elements of the dual space of smooth functions. In particular, a (1,1) current on a complex manifold has the

local form $\sum T_{ij} \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_i}$ where the T_{ij} are distributions. A positive current of this type

is characterized by the T 's being measures. The d -operator operates on currents by adjoint action, just as derivatives operate on distributions, so $dC(\text{form}) =$ (by definition) $C(d(\text{form}))$. Thus d lowers "dimension" of currents. Currents split into types, just as forms do, on a complex manifold.] The main result is a characterization of Kähler manifolds. Namely, they prove:

A compact complex manifold M is Kähler (that is, admits a Kähler metric) if and only if M admits no positive current of type $(1,1)$ which is the $(1,1)$ part of a boundary, that is, it is the $(1,1)$ part of the operator d applied to another current.

The idea of the proof is as follows: Consider the set of forms of type $(1,1)$ that are positive at each point, that is, the associated Hermitian form is positive definite at each point of M . This is a (real) convex (half) cone in the space of $(1,1)$ forms, considered as a real vector space. If one of these forms is closed, then M has a Kähler metric (and conversely), namely the positive definite Hermitian form associated is a Kähler metric. If none of them is closed, then an argument of Hahn-Banach type shows that there is a (closed) hyperplane in the space of forms which contains the kernel of d , but has empty intersection with the (half) cone. Then $\pm a$ linear functional that has its kernel=this hyperplane is a current that violates the hypothesis of the theorem; such a linear functional vanishes on kernel d is thus necessarily in the image of d acting on currents, that is, the adjoint of d acting on forms, made an operator on currents as already defined.

(This type of argument will reappear later when we discuss balanced metrics.)

This is in effect a current version of the fact that on a Kähler manifold, a complex submanifold cannot be homologous to 0. [Logic: If N were the boundary of N' then integral over N of the appropriate power of the Kähler form would be the integral over N' of d of that power of the Kähler form and would hence be 0. But then N has volume 0 and is thus not a submanifold.] On non-Kähler manifolds, it can happen that a complex submanifold can be homologous to 0. For example in the famous Calabi-Eckmann complex structures on the products of odd dimensional spheres, there are complex tori that are homologous to 0. [These come from the fibration of each factor with fibre a circle, the fibration being the projection of each odd sphere onto the corresponding complex projective space. The complex structure arises from thinking of the tori as Riemann surfaces and combining this with the complex structures from the complex projective spaces. In this set up, the tori are by definition complex submanifolds but they are homologous to 0.] The Harvey –Lawson characterization takes care of the elliptic surface case: namely, that they are Kähler if and only if they have even b_1 .

What is left is the situation of K3 surfaces. Here all the surfaces have b_1 even. So one wants to show that all of them are Kähler. It was shown by Kodaira that they had Kähler deformations, even arbitrarily close Kähler manifolds. Now by Yau's solution of the 0 case of the Calabi Conjecture, there are Ricci flat Kähler metrics on each of these Kähler K3s. but the metric depends on a choice of "polarization" (choice of Kähler class). Todorov and Siu showed that one could control the Kähler class of the approximating Kähler K3s so that there was a limit Kähler metric on the original K3. (More on this later including Hitchin idea of three different complex structures obtained from the metric).

Thus in all cases, any compact complex surface, b_1 being even implies there is a Kähler metric (both for elliptic surfaces and K3 surfaces, the only cases in question really).

Another idea: algebraic dimension (and class VII surfaces)

Consider the field of meromorphic functions (global functions that are locally the quotient of two holomorphic functions). On a compact complex manifold this is finitely generated as a field over the complex numbers, and it has transcendence degree less than or equal to the complex dimension of the manifold (theorem of C.L. Siegel). For complex surfaces, it is thus at most 2. This transcendence degree is called the "algebraic dimension".

This turns out to be central for considering non-Kähler complex surfaces, the somewhat mysterious class VII surfaces. In particular, if the algebraic dimension of one of these is 1, then it is an elliptic surface. If the algebraic dimension is 0 and $b_2 = 0$ and S contains curves, then it is a Hopf surface (that is surface with universal cover $\mathbb{C}^2 - \{(0,0)\}$), as shown by Kodaira. Thus the remaining cases are

(i) $b_2 = 0$, no curve

and

(ii) $b_2 > 0$.

The first of these two cases is illustrated by examples constructed by Inoue (these will be described in detail next time). They are quotients of $H \times \mathbb{C}$, where H is the upper half plane in the complex numbers \mathbb{C} . Inoue proved that any example of the type of the first case here was one of his example provided that there is a line bundle L on the surface such that there are non trivial holomorphic sections of the tensor product of the tangent bundle of the surface with L . Bogomolov argued that this always happened: such a bundle L always exists. Then Inoue's examples are the only examples. Bogomolov's argument is complicated and hard to follow. Li, Yau, Zheng (Illinois Journal 34(1990)) gave a shorter argument using Hermitian Yang Mills results of Li, Yau (in Mathematical Aspects of String Theory, World Scientific, 1987). These topics will be treated in more detail next time.

Robert E. Greene